Diffusion in the special theory of relativity

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The Markovian diffusion theory is generalized within the framework of the special theory of relativity. Since the velocity space in relativity is a hyperboloid, the mathematical stochastic calculus on Riemanian manifolds can be applied but adopted here to the velocity space. A generalized Langevin equation in the fiber space of position, velocity, and orthonormal velocity frames is defined from which the generalized relativistic Kramers equation in the phase space in external force fields is derived. The obtained diffusion equation is invariant under Lorentz transformations and its stationary solution is given by the Jüttner distribution. Besides, a nonstationary analytical solution is derived for the example of force-free relativistic diffusion.

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I. INTRODUCTION

The formulation of a consistent theory of Markovian diffusion within the framework of the relativity theory is a long-standing problem in physics. Over the years, numerous studies have been devoted to this issue (see, e.g., [1–13]) with different and apparently irreconcilable points of view. At least up to the knowledge of the author, a generally accepted consistent solution of this problem is still missing. However, besides its fundamental theoretical interest, such theory is of particular importance in several applications such as in high-energy collision experiments (see, e.g., [14]), astrophysics (see, e.g., [15]), and others.

An alternative physical approach for the description of a relativistic gas in a heat bath is given by statistical thermodynamics and the Boltzmann equation. Jüttner derived the thermal equilibrium distribution for a relativistic gas already in 1911 [16]. After that, many authors have given important contributions to the development of the relativistic kinetic theory (for an introduction, see, e.g., [17,18]). In spite of this progress, in recent years a controversial debate about the correct generalization of Maxwell's velocity distribution in special relativity arose [8,19–21]. Recently, numerical microscopic one-dimensional simulations [22] and a critical analysis of alternative findings [23] yield arguments in favor of the Jüttner distribution. Lately, a comprehensive review of relativistic diffusion processes has been published [13]. Besides the issue of the stochastic relativistic diffusion theory, it also includes relativistic equilibrium thermostatics and microscopic models for Langevin-type equations and a more complete list of references.

Diffusion theory in the Euclidean space R^d is a well-developed topic (see, e.g., [24,25]). However, the description of diffusion on non-Euclidean manifolds M^d is a subject containing several pitfalls. There exists a well-developed rigorous mathematical theory of stochastic differential equations and diffusion processes on Riemannian manifolds with a definite metric signature (see, e.g., [26,27]). The stochastic calculus on Riemannian manifolds found considerable interest in mathematics and has played a central role in recent years within the analysis in path and loop spaces in topology and other fields. However, this mathematical approach cannot be applied to describe diffusion on Minkowski or pseudo-Riemannian manifolds with indefinite metrics. In the present paper, we derive a physically motivated modification of this calculus to describe diffusion in the phase space of position and velocity, in which the difficulties in the description of diffusion on manifolds with indefinite metric signature are bypassed.

The main aim of the paper is the derivation of a relativistic diffusion equation in phase space, which generalizes the nonrelativistic Kramers equation. A crucial factor in the derivation is the observation that the velocity space in relativity is a hyperboloid (or a special three-dimensional Riemannian manifold) embedded into the velocity Minkowski space. This requires the application of the stochastic calculus on Riemanian manifolds adopted to the velocity space. Correspondingly, in this approach a relativistic stochastic differential equation is defined, which generalizes the Langevin equation and introduces a moving velocity frame necessary for a consistent treatment. As will be shown, the derived relativistic diffusion equation satisfies the general principle of special relativity and is invariant under Lorentz transformations. The steady-state solution of this equation for a heath bath with constant friction coefficient yields the Jüttner distribution.

The paper is organized as follows. In Sec. II, the general concept of the mathematical stochastic calculus on Riemannian manifolds is briefly described. In Sec. III,, a generalized relativistic Langevin equation in the fiber bundle of position, velocity, and orthonormal velocity frames is defined and the relativistic diffusion equation for the probability density function or the transition probability is derived. In Sec. IV, the steady-state solution for a relativistic gas in a heat bath with constant friction coefficient and, in Sec. V, the nonsteady solution for the force-free case is derived; in Sec. VI, the conclusions are presented. In the Appendix, basic formulas and theorems of the stochastic calculus on Euclidean and Riemannian manifolds are summarized, which are required for an understanding of the paper and fix the notations.

II. MATHEMATICAL STOCHASTIC CALCULUS ON RIEMANNIAN MANIFOLDS

Stochastic differential equations in diffusion theory in a d-dimensional Euclidean space R^d with continuous pathway

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are defined by the fundamental d-dimensional Wiener process $W^a(t)$. On a Riemannian manifold M^d, the fundamental Wiener process is difficult to handle. By using an inadequately posed formulation of a stochastic differential equation, it is not assured that its solution remains on the manifold M^d, which leads to inconsistent results. The key idea in the mathematical concept of diffusion on general d-dimensional Riemannian manifolds M^d (with definite metric signature) is to define a stochastic process on the curved manifold using the fundamental Wiener process, each component of which is a process in the Euclidean space R^d [26,27]. Intuitively, we can understand this concept as follows. Consider a two-dimensional stochastic motion of a particle on a plane. If the trajectory of the particle is traced in ink and a sphere on the plane is rolled along the stochastic curve without slipping the resulting transferred path defines a random curve or a stochastic Markovian process on the sphere. This method can be applied for diffusion on a general Riemannian manifold. The tangent space of a Riemannian manifold is endowed with Euclidean structure and, therefore, we can move the manifold in the tangent space by construction of a parallel translation along the stochastic curve with the help of the orthonormal frame vectors $e_a = e_a^i(\mathbf{x})\partial_i$ $(i,a=1,\ldots,d)$ and the Christoffel connection coefficients Γ^j_{ib}, $\mathbf{x}=(x_1,\ldots,x_d)$, $\partial_i=\partial/\partial x^i$. In local coordinates on a Riemannian manifold, the infinitesimal motion of a smooth curve $c^i(t)$ in M^d is that of $\gamma^i(t)$ in the tangent space (which can be identified with R^d) by using a parallel transformation: $dc^i = e^i_a(\mathbf{x})d\gamma^a$ and $de^i_a(\mathbf{x}) = -\Gamma^i_{ml}e^l_a dc^m$. Therefore, a random curve can be defined in the same way by using the canonical realization of a d-dimensional Wiener process (defined in the Euclidean space) and substituting $d\gamma^a \to dW^a(t)$. Thus, the stochastic differential equations describing diffusion on a Riemannian manifold in the orthonormal frame bundle $O(M)$ with coordinates $O(M)=\{x^i, e^i_a\}$ are given by

$$dx^i(\tau) = e^i_a(\tau)\circ dW^a_\tau + b^i(\tau)d\tau,$$

$$de^i_a(\tau) = -\Gamma^i_{ml}e^l_a \circ dx^m(\tau). \tag{1}$$

Here $\delta^{ab}e^i_a[\mathbf{x}(\tau)]e^j_b[\mathbf{x}(\tau)] = g^{ij}$, $\partial_i e^j_a = -\Gamma^j_{ik}e^k_a$, g^{ij} is the Riemannian metric, and δ^{ab} is the flat Euclidean metric, where δ^{ab} is the Kronecker symbol. The components of the elementary Wiener process $dW^a = W^a(t+\Delta t) - W^a(t)$ are defined in the Euclidean space with the probability density $\mathrm{P}(W^a) = (2D\pi\Delta t)^{-1/2}\exp(-\frac{[W^a(t)]^2}{2D\Delta t})$ and with the expectation values $\langle W^a\rangle=0$ and $\langle W^a(\tau)W^b(\tau+s)\rangle = Ds\,\delta_{ab}$. b^i's are the components of an arbitrary tangential vector and D is the diffusion constant, which here is independent on the time and space variables. Equation (1) is defined in the Stratonovich calculus (denoted by the symbol $\circ$).

Associated to each diffusion process, there is a second-order differential operator denoted as the generator $\mathbf{A}$ of the diffusion. This operator is associated with the Kolmogorov backward equation and is defined in the Appendix for diffusion processes on Euclidean and Riemannian manifolds, respectively. Corresponding to Eqs. (A16) and (A17), the diffusion generator $\mathbf{A}_{O(M)}$ on $O(M)$ can be projected on M^d with $f(\mathbf{r}) = f(\mathbf{x},\mathbf{0})$, $\mathbf{r}=(x^i, e^i_j)$ using the relation $\mathbf{A}_{O(M)}f(\mathbf{r}) = \mathbf{A}_M f(\mathbf{x})$, where

$$\mathbf{A}_M = \frac{D}{2}\delta^{ab}(e^i_a\partial_i e^j_b\partial_j) + b^i\partial_i = \left(\frac{D}{2}\Delta_M + b^i\partial_i\right) \tag{2}$$

and $\Delta_M = g^{ij}\partial_i\partial_j - g^{ij}\Gamma^k_{ij}\partial_k$ is the Laplace-Beltrami operator on the manifold M^d. The generalized Fokker-Planck equation is obtained by the adjoint of the diffusion generator $\mathbf{A}^*_M$ (which includes the volume element $\sqrt{g}$, $g=\det\{g^{ij}\}$). Since the Laplace-Beltrami operator is self-adjoint, $\Delta_M = \Delta^*_M$, this equation takes the form

$$\frac{\partial\Phi}{\partial\tau} = -\,\mathrm{div}_x(\mathbf{b}\Phi) + \frac{D}{2}\Delta_M\Phi, \tag{3}$$

where $\mathrm{div}_x(\mathbf{b}\Phi) = g^{-1/2}\,\partial_i(g^{1/2}b^i\Phi)$ is the divergence operator on the Riemannian manifold, $\Phi=\Phi(\mathbf{x},\tau|\mathbf{y},0)$ is the transition probability with the initial condition $\Phi(\mathbf{x},0|\mathbf{y},0) = \delta(\mathbf{x}\text{-}\mathbf{y})$ and adequate boundary conditions at infinity. The probability density $\varphi(x^i,\tau)$ is determined by the same equation with the initial condition $\varphi(\mathbf{x},\tau=0) = \varphi^0(\mathbf{x})$.

A remarkable feature of Markovian diffusion on a Riemannian manifold is the supposition that for the diffusion coefficients in Eq. (1) only the orthonormal frame coefficients $e^i_a(\mathbf{x})$ are admissible, which are directly related to the geometry of the Riemannian manifold. In contrast, on Euclidean manifolds, a much more general class of diffusion coefficients are permitted.

III. RELATIVISTIC DIFFUSION

A direct application of the mathematical calculus of diffusion processes on general Riemannian manifolds for relativistic physics is not possible due to the supposition restricting the stochastic formalism to the special case of a Riemannian manifold with a definite metric signature, while in relativity theory the Minkowski or pseudo-Riemannian manifolds exhibit an indefinite metric signature $(-,+,+,+)$. Moreover, it has been proven that Markovian diffusion processes in the base manifold (position space) on a Minkowski or pseudo-Riemannian manifold do not exist [3,4]. However, considering more carefully the mathematical model described by Eq. (1), one can recognize a significant difference to the physical nonrelativistic diffusion model described by the Langevin equation. In Eq. (1), the noise term directly acts on the position variable in the base space, while the noise term in the Langevin equation operates like a force on the change in the velocity in the tangent space. This crucial difference in the mathematical model to the physically motivated Langevin approach enables a generalization of the Markovian diffusion theory within the framework of the special relativity theory performed in the phase space of coordinates x^i and the spatial components of the four velocity u^i with $(i=1,2,3)$.

The four velocity in special relativity defined by $u^\mu = dx^\mu/d\tau$, $(\mu=0,1,2,3)$ is a hyperboloid (or pseudosphere) described by the relation

$$(u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2 = 1, \quad (4)$$

where τ is the proper time with $d\tau = -\eta_{\mu\nu}dx^\mu dx^\nu$, $\eta_{\mu\nu}$ is the Minkowski metric. This means that the relativistic velocity space is a noncompact hyperbolic three-dimensional Riemannian manifold (and not pseudo-Riemannian) embedded into the four-dimensional Minkowski velocity space. Therefore, with an appropriate modification, we can apply for the velocity space the mathematical stochastic calculus for Riemannian manifolds as presented in Sec. I and in the Appendix. Since the stochastic force acts directly only on the change in the velocity and not on the position coordinates, we can define the relativistic generalization of the Langevin equation in a fiber bundle here denoted by $F(M_L)$ with $\{x^i, u^i, E^i_a\} = F(M_L)$, where x^i belongs to the Lorentzian base manifold M_L, the relativistic velocity u^i to the tangent space TM_L and $E^i_a(u)$ are the moving orthonormal frames in the hyperbolic velocity space. Locally, this fiber bundle is simply the product space of these three subspaces. With a corresponding modification of Eq. (1), the generalized relativistic Langevin equations can be defined in the fiber bundle space $F(M_L)$ by

$$dx^i(\tau) = u^i(\tau)d\tau,$$

$$du^i(\tau) = E^i_a(\tau) \circ dW^a(\tau) + F^i(\tau)d\tau,$$

$$\begin{aligned} dE^i_a(\tau) &= -\gamma^j_{ml}(\mathbf{u})E^l_a(\tau) \circ du^m(\tau) \\ &= -\gamma^j_{ml}(\mathbf{u})E^l_a(\tau)[E^m_b(\tau) \circ dW^b(\tau) + F^m(\tau)d\tau]. \end{aligned} \quad (5)$$

Here τ is an evolution parameter along the world lines of the particles, which can be chosen as the proper time. The laboratory time $t = \tau u^0/c$ is a function of the proper time τ and u^0, which here and below is defined by $u^0 = [1 + (u^1)^2 + (u^2)^2 + (u^3)^2]^{1/2}$. $\gamma^j_{ml}(\mathbf{u})$ are the Christoffel connection coefficients on the hyperboloid and $F^i = K^i/m$, where K^i are the spatial components of the four force, m is the rest mass of the particles, and the indices a,b denote the spatial components in the hyperbolic space $(a,b = 1,2,3)$. Since the stochastic term $dW^a(\tau)$ does not act directly on the position variable $x^i(\tau)$, the indefinite signature of the Lorentzian manifold here does not create any difficulty, as it arise for a stochastic differential equation as Eq. (1), but for a manifold with indefinite metric. The stochastic force is described as above by the fundamental Wiener process with $\langle W^a \rangle = 0$ and the correlator $\langle W^a(\tau)W^b(\tau+s)\rangle = Ds\,\delta_{ab}$ is defined by an empirical diffusion constant D, which here is independent on the velocity. Sufficient conditions for the existence and uniqueness of the stochastic differential Eqs. (5) are that the drift and diffusion coefficients satisfy the uniform Lipschitz condition and the stochastic process $\mathbf{X}(\tau) = \{\mathbf{x}(\tau), \mathbf{u}(\tau)\}$ is adapted to the Wiener process $\mathrm{W}^a(\tau)$, that is, the output $\mathrm{X}(\tau_2)$ is a function of $\mathrm{W}^a(\tau_1)$ up to that time $(\tau_1 \le \tau_2)$ [25]. The moving frames in the hyperbolic velocity space are defined by the relation

$$\delta^{ab}E^i_a E^j_b = G^{ij}, \quad (6)$$

or equivalently,

$$G_{ij}E^i_a E^j_b = \delta_{ab}, \quad (7)$$

where G_{ij} is the Riemannian metric of the hyperbolic velocity space, G^{ij} is the inverse matrix of G_{ij}, and the Christoffel connection coefficients $\gamma^j_{ml}(\mathbf{u})$ on the hyperboloid are given by

$$\gamma^j_{jk}(\mathbf{u}) = \frac{1}{2}G^{im}[\partial G_{jm}/\partial u^k + \partial G_{mk}/\partial u^j - \partial G_{jk}/\partial u^m]. \quad (8)$$

Since the manifold on the hyperboloid is embedded into the Minkowski space, the metric $G_{ij}(u)$ can be calculated from the infinitesimal arc length given by $ds^2_u = -(du^o)^2 + (du^1)^2 + (du^2)^2 + (du^3)^2$ with $u^0 = [1 + (u^1)^2 + (u^2)^2 + (u^3)^2]^{1/2}$. In this way, we obtain $ds^2_u = G_{ij}(u)du^i du^j$ with $G_{ij}(\mathbf{u}) = \delta_{ij} - (u^i u^j)/(u^0)^2$, $G = \det G_{ij} = (u^0)^{-2}$, and $\gamma^j_{jk}(\mathbf{u}) = -u^i G_{jk}$. Corresponding the definition of fundamental vector fields on $O(M)$ in the Appendix, one can now introduce the fundamental horizontal vector fields H_a and H_0 on the fiber bundle $F(M_L)$. With corresponding modifications, we find analogous as in Eq. (A16),

$$H_a = E^i_a \frac{\partial}{\partial u^i} - \gamma^j_{ml}(\mathbf{u})E^l_a E^m_b \frac{\partial}{\partial E^i_b},$$

$$H_0 = u^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial u^i} - \gamma^j_{ml}(\mathbf{u})E^l_a(\tau)F^m \frac{\partial}{\partial E^i_a}. \quad (9)$$

The diffusion operator $\mathbf{A}_{F(M_L)}$ for the stochastic process is given, as in Eq. (A17), by

$$\mathbf{A}_{F(M_L)} = \frac{D}{2}\delta^{ab}H_a H_b + H_0. \quad (10)$$

We project the stochastic curve from the fiber space $F(M_L)$ with coordinates $\mathbf{r} = \{x^i, u^i, E^i_a\}$ to the phase space with coordinates $\{x^i, u^i\}$: $\mathbf{A}_{F(M_L)}f(\mathbf{r}) = \mathbf{A}_P f(\mathbf{x}, \mathbf{u}, \mathbf{0})$, where the diffusion generator in the phase space $\mathbf{A}_P$ is given by

$$\mathbf{A}_P = \frac{D}{2}\delta^{ab}E^i_a \frac{\partial}{\partial u^i}E^j_b \frac{\partial}{\partial u^j} + u^i\,\partial/\partial x^i + F^i\,\partial/\partial u^i. \quad (11)$$

The special-relativistic diffusion equation in the phase space is given by the adjoint of the operator $\mathbf{A}_P$ and analogous to Eq. (A20) the generalized relativistic Kramers equation takes the form

$$\frac{\partial \Phi}{\partial \tau} = -u^i \frac{\partial \Phi}{\partial x^i} - \mathrm{div}_u(\mathbf{F}\Phi) + \frac{D}{2}\Delta_u \Phi, \quad (12)$$

where Δ_u is the Laplace-Beltrami operator of the hyperbolic velocity space given by

$$\Delta_u = G^{ij}\frac{\partial^2}{\partial u^i\,\partial u^j} - G^{ij}\gamma^k_{ij}\frac{\partial}{\partial u^k} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^i}\left(\sqrt{G}G^{ij}\frac{\partial}{\partial u^j}\right), \quad (13)$$

and the corresponding divergence operator by

$$\mathrm{div}_u(\mathbf{F}\Phi) = \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^i}(\sqrt{G}F^i\Phi), \quad (14)$$

with $G = \det\{G_{ij}\}$ and $G^{ij} = \delta^{ij} + u^i u^j$.

Equation (12) represents the relativistic generalization of the Kramers equation for the probability density function $\Phi=\phi(\tau;\mathbf{x},\mathbf{u})$ with the initial condition $\phi(\tau=0;\mathbf{x},\mathbf{u})=\phi_0(\mathbf{x},\mathbf{u})$. The transition probability is determined by the same equation but is defined by the initial condition

$$\Phi(\mathbf{x},\mathbf{u},\mathbf{0}|\mathbf{x}_0,\mathbf{u}_0,0)=\tfrac{1}{\sqrt{G}}\delta(u^1-u_0^1)\delta(u^2-u_0^2)\delta(u^3-u_0^3)\delta(x_1-x_1^0)\delta(x_2-x_2^0)\delta(x_3-x_3^0).$$

If the force F^i depends on the time t, we have to substitute t by $t=\tau[1+(u^1)^2+(u^2)^2+(u^3)^2]^{1/2}/c$. For an external electromagnetic field $F^{\mu\nu}$, the normalized force F^i is given by $F^i=eF^i_{\nu}u^{\nu}$. For small velocities ($|u^i|^2\ll 1$), Eq. (12) pass over to the nonrelativistic Kramers equation [28].

In the relativistic framework, the Lorentz invariance of the physical laws is one of the most fundamental property. Equation (12) refers to a special inertial rest frame Σ of an observer. Now consider a second observer at rest in another inertial frame Σ' that moves with constant velocity w relative to Σ. There exist no doubt and strong arguments in the literature that the probability density function $\phi(\tau;\mathbf{x},\mathbf{u})$ has to be a scalar in the phase space to be consistent within the relativistic framework, i.e., it fulfills the condition

$$\phi'(\tau',\mathbf{x}',\mathbf{u}')=\phi(\tau,\mathbf{x},\mathbf{u}), \tag{15}$$

where the transformed variables

$$x'^i=\Lambda^i_j x^j+\Lambda^i_0 x^0,\quad u'^j=\Lambda^i_j u^j+\Lambda^i_0 u^0,\quad \tau'=\tau \tag{16}$$

are related by the Lorentz transformation. Within the approach of relativistic statistical physics, Eq. (15) was proven by van Kampen [29] for the one-particle distribution function. Note that before, several authors argued that the relation (15) can be proven by the Lorentz invariance of the phase-space volume element $d^3x d^3u$ (see, e.g., [18]). But as discussed in Refs. [29,30], this argument ignores the fact that the observations in Σ and Σ' refer to different hypersurfaces; it belongs in Σ to the hypersurface x^0=const, but in Σ' it does not belong to the hypersurface x'^0=const. Note that in contrast, the particle density and the current density (i.e., the integrals over the velocities) transform like a four vector. The Lorentz invariance of $\phi(\tau;\mathbf{x},\mathbf{u})$ requires that Eq. (12) is also invariant with respect to a Lorentz transformation. By using $x^0=\tau u^0$, the chain rule $\partial/\partial x^i=\Lambda^i_j\partial/\partial x'^j$ and the inverse transformation $u^i=\bar{\Lambda}^i_j u'^j+\bar{\Delta}^i_0 u'^0$ with $\bar{\Lambda}^{\alpha}_{\beta}\Lambda^{\beta}_{\gamma}=\delta^{\alpha}_{\gamma}$, we find

$$u^i\frac{\partial\phi}{\partial x^i}=u'^i\frac{\partial\phi}{\partial x'^i}-u^0\Lambda^i_0\frac{\partial\phi}{\partial x'^i}$$

$$\frac{\partial\phi}{\partial\tau}=\frac{\partial\phi}{\partial\tau'}+u^0\Lambda^i_0\frac{\partial\phi}{\partial x'^i}. \tag{17}$$

On a Riemannian manifold, the divergence operator and the Laplace-Beltrami operator are intrinsically invariant with respect to general coordinate transformations. Therefore, Lorentz transformation on the hyperboloid does not change the explicit form of these operators. For the divergence operator, this can be simply proven by the transformation property of the covariant differentiation $D_jF^i\equiv\partial F^i/\partial u_j+\gamma^j_{jk}F^k$ given by $D'_jF'^i=(\partial u'^i/\partial u^k)(\partial u^l/\partial u'^j)D_lF^k$. For the divergence operator $\mathrm{div}_u(\mathbf{F}\phi)=D_j(F^j\phi)$, this yields the relation $D'_jF'^j=D_jF^j$. The invariance of the Laplace-Beltrami operator $\Delta_u\phi=D_j(G^{ij}\partial_i\phi)$ can be similarly proven. Since $G^{ij}\partial_i\phi=A^j$ transforms like a vector and the divergence D_jA^j is as shown above an invariant operation, the relation $\Delta_u\phi=\Delta_{u'}\phi$ follows. In the same way, the invariance of the infinitesimal arc length ds_u^2 is proved. By using Eq. (17), we express the variables x^i, u^j by the new variables x'^i, u'^j in the inertial frame Σ' and account the invariance of the divergence and Laplace-Beltrami operator. Then Eq. (12) takes the form

$$\frac{\partial\phi}{\partial\tau'}=-u'^i\frac{\partial\phi}{\partial x'^i}-\mathrm{div}_{u'}(\mathbf{F}'\phi)+\frac{D}{2}\Delta_{u'}\phi. \tag{18}$$

Thus, the derived relativistic diffusion equation satisfies the general principle of special relativity and is invariant under Lorentz transformations.

Note that Eq. (12) differs from previously derived relativistic diffusion equations. Debbasch and Rivet [6] introduced a phenomenological relativistic Langevin equation in the phase space and derived from this a generalized Kramers equation of the classical Ornstein-Uhlenbeck process, in which the diffusion term is given by the three-dimensional Euclidean-Laplacian in the momentum space. The difference of this result to Eq. (12) is explained by the fact that in [6] the Wiener process is described in the same way as in a Euclidean space; but the velocity space is a special Riemannian manifold, which requires to use a rigorous stochastic calculus on non-Euclidean manifolds. As explained in Sec. II and above the introduction of a moving velocity frame $E^i_a(u)$ in the relativistic Langevin equations (5) is the crucial point, which avoids the difficulties in the description of the Wiener process on Riemannian manifolds. If we ignore this problem and substitute $E^i_a(u)$ by the Kroneker symbol, the diffusion term in Eq. (12) passes into the corresponding diffusion term of Ref. [6]. A different more general approach was presented by Dunkel and Hänggi [8,13], but the problem to handle the Wiener process on the hyperbolic velocity space in a rigorous way was also not achieved and the derived diffusion equation as well differs from Eq. (12).

The relativistic diffusion process considered up to now is parametrized in terms of the proper time τ. But it is a matter of convenience to parametrize this process alternatively in terms of the time x^0 of the inertial frame of the observer. The derivation of the diffusion equation in this parametrization can be performed using the mathematical theorem of random time change in stochastic differential equations (see, e.g., [25,26]). The proper time τ is related with x^0 by $d\tau=(u^0)^{-1}dx^0$, depending on the stochastic variables u^i. This means that the transformation to x^0 is a random time transformation. Therefore, the time change in an Ito integral is again an Ito integral but driven by a different Wiener process $d\tilde{W}(x^0)=dW(\tau)(u^0)^{1/2}$ [25,26]. This rule for a random time change refers to the Ito interpretation, but one can use the relation (A9) for the change in the drift and diffusion coefficients if an Ito integral is transformed into a Stratonovich integral. Using $d\tau=(u^0)^{-1}dx^0$, the relativistic Langevin equations (5) can be rewritten in the parametrization with x^0 as follows:

$$dx^i(x^0) = u^i(\tau)(u^0)^{-1}dx^0,$$

$$du^i(x^0) = E^i_a(x^0)(u^0)^{-1/2} \circ d\widetilde{W}(x^0) + F^i(u^0)^{-1}dx^0 - \frac{D}{2}\delta^{ab}E^i_a(x^0)E^j_b(x^0)(u^0)^{-1/2} \times \frac{\partial}{\partial u^j}[(u^0)^{-1/2}]dx^0,$$

$$dE^i_a(x^0) = -\gamma^j_{ml}(\mathbf{u})E^l_a(x^0) \circ du^m(x^0). \tag{19}$$

The generalized relativistic Kramers equation in the time parametrization x^0 of the observer inertial frame therefore can be derived analogous as above and leads to the following result:

$$u^0\frac{\partial\Phi}{\partial x^0} = -u^i\frac{\partial\Phi}{\partial x^i} - \mathrm{div}_u(\mathbf{F}\Phi) + \frac{D}{2}\Delta_u\Phi. \tag{20}$$

As seen analogous to the relativistic Boltzmann equation, the left-hand side and the first term o.r.s. of Eq. (20) can be identified with the covariant expression $u^\mu\partial/\partial x^\mu$, while the other terms are identical with corresponding terms in Eq. (12).

For the solution of the relativistic diffusion equation, it is convenient to introduce the hyperbolic coordinate system for the four velocity defined by $u^1=\mathrm{sh}\alpha \sin\vartheta \cos\varphi$, $u^2=\mathrm{sh}\alpha \sin\vartheta \sin\varphi$, $u^3=\mathrm{sh}\alpha \cos\vartheta$, and $u^0=\mathrm{ch}\alpha$. We denote the velocities in the non-Cartesian coordinates by $\bar{u}^1=\alpha$, $\bar{u}^2=\vartheta$, $\bar{u}^3=\varphi$, and $a=1,2,3$. The metrics in this coordinates are simply to calculate and are given by $G_{11}=1$, $G_{22}=\mathrm{sh}^2\alpha$, $G_{33}=\mathrm{sh}^2\alpha\sin^2\vartheta$, and $G_{ij}=0$ for $i\neq j$. With the given metric, the Laplace-Beltrami operator Δ_u in the hyperbolic velocity space takes the form

$$\Delta = \frac{\partial^2}{\partial\alpha^2} + 2\mathrm{cth}\alpha\frac{\partial}{\partial\alpha} - \frac{1}{(\mathrm{sh}\alpha)^2}\left(\frac{\partial^2}{\partial\vartheta^2} + \mathrm{ctg}\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{(\sin\vartheta)^2}\frac{\partial^2}{\partial\varphi^2}\right) \tag{21}$$

and

$$\mathrm{div}_u(\mathbf{F}\Phi) = (\mathrm{sh}\alpha)^{-2}\frac{\partial}{\partial\alpha}[(\mathrm{sh}\alpha)^2F^a\Phi] - (\mathrm{sh}\alpha)^{-1}(\sin\vartheta)^{-1}\frac{\partial}{\partial\vartheta}(\sin\vartheta F^\vartheta\Phi) - (\mathrm{sh}\alpha)^{-1}(\sin\vartheta)^{-1}\frac{\partial}{\partial\varphi}(F^\varphi\Phi) \tag{22}$$

is the divergence operator in the hyperbolic velocity space. Here the force in the hyperbolic coordinate system F^a, F^ϑ, F^φ is related with F^i by $F^a=(\mathrm{ch}\alpha)^{-1}[\sin\vartheta(\cos\varphi F^1+\sin\varphi F^2)+\cos\vartheta F^3]$, $F^\vartheta=(\mathrm{sh}\alpha)^{-1}[\cos\vartheta(\cos\varphi F^1+\sin\varphi F^2)-\sin\vartheta F^3]$, and $F^\varphi=(\mathrm{sh}\alpha)^{-1}(\sin\vartheta)^{-1}[-\sin\varphi F^1+\cos\varphi F^2]$. The probability density is determined by the initial condition $\phi(\tau=0;x_i,\alpha,\vartheta,\varphi)=\phi_0(x_i,\alpha,\vartheta,\varphi)$ and the transition probability by

$$\Phi(\mathbf{x},\alpha,\vartheta,\varphi,0|\mathbf{x}_0,\alpha_0,\vartheta_0,\varphi_0,0) = (\mathrm{sh}\alpha)^{-2}(\sin\vartheta)^{-1}\delta(\alpha-\alpha_0)\delta(\vartheta-\vartheta_0)\delta(\varphi-\varphi_0) \times\delta(x_1-x_1^0)\delta(x_2-x_2^0)\delta(x_3-x_3^0).$$

IV. STEADY-STATE SOLUTION OF PARTICLES IN A HEAT BATH: THE JÜTTNER DISTRIBUTION

First, we consider particles with a rest mass m of a gas in an isotropic homogenous heat bath. The interaction of particles with the bath is described by a random noise force and a friction force. In the nonrelativistic theory, the friction force is given by $f^i=-\nu m v^i$, where ν is the friction coefficient and v^i are the components of the nonrelativistic velocity. The relativistic generalization of the friction force requires the introduction of a friction tensor ν^i_α similar to the pressure tensor in special relativity theory [6,8]. The friction force is expressed as $F^i=\nu^i_\alpha[u^\alpha-U^\alpha]$, where U^α is the four velocity of the heat bath. For an isotropic homogeneous heat bath, the friction tensor is given by

$$\nu^i_\alpha = \nu(\eta^i_\alpha + u^iu_\alpha), \tag{23}$$

with ν denoting the scalar friction coefficient measured in the rest frame of the particles. In the laboratory frame, the heat bath is at rest described by $U^\alpha=(1,0,0,0)$. Therefore, the friction force is given by $F^i=-\nu u^iu^0$ or in hyperbolic coordinates $F^a=-\nu\mathrm{sh}\alpha$, $F^\vartheta=F^\varphi=0$. We consider the spatial homogenous and isotropic solution of Eq. (12) with Eqs. (13), (14), and (23) described by

$$\frac{\partial\phi(a)}{\partial\tau} = \frac{D}{2}\left[\frac{\partial^2}{\partial\alpha^2} + 2\mathrm{cth}\alpha\frac{\partial}{\partial\alpha}\right]\phi(a) + \nu(\mathrm{sh}\alpha)^{-2}\frac{\partial}{\partial\alpha}[(\mathrm{sh}\alpha)^3\phi(\alpha)]. \tag{24}$$

The steady-state solution of this equation is given by $\phi(\alpha)=C\exp\{-\chi\mathrm{ch}\alpha\}$ with $\chi=2\nu/D$ and $C=4\pi K_2(\chi)/\chi$. $K_2(\chi)$ denotes the modified Hankel function. This distribution is identical with the Jüttner equilibrium distribution if we use the relation $\chi=\frac{c^2m}{kT}$, where T is the temperature of the heath bath and k is the Boltzmann constant. Consequently, we find for the diffusion constant $D=2\nu kT/mc^2$. The above-derived relativistic diffusion Eq. (12) yields for the three-dimensional case the correct thermodynamic relativistic equilibrium distribution for a constant friction coefficient. Note that in previously derived relativistic diffusion equations [6,8], the Jüttner equilibrium distribution for a relativist gas only arises as the steady-state solution for a specifically adapted energy-dependent friction constant $\nu=\nu(u^0)$. Recently, fully relativistic one-dimensional molecular-dynamics simulations favored the Jüttner distribution in the one-dimensional case [22]. Besides an independent support of this distribution is the kinetic theory based on the relativistic Boltzmann equation, which yields as the only distribution function, which implies a vanishing of the collision term in equilibrium the Jüttner distribution (see, e.g., [17,18]).

V. NONSTEADY SOLUTION FOR THE FORCE-FREE CASE

Now we consider the nonsteady solution of Eq. (12) for a spatial homogenous gas with vanishing force F^i. We use the Laplace transformation $\Phi(\alpha,\vartheta,\varphi,\tau)=\int_0^\infty\tilde{\Phi}(\alpha,\vartheta,\varphi,\lambda)\times\exp(-\lambda\tau)d\lambda$ and for the eigenvalue functions the ansatz $\tilde{\Phi}_M^J(\alpha,\vartheta,\varphi,\lambda)=g_J^\lambda(\alpha)Y_M^J(\theta,\varphi)$, where $Y_M^J(\theta,\varphi)=\mathrm{P}_M^J(\vartheta)e^{iM\varphi}$ are the spherical harmonics with the associated Legendre functions $\mathrm{P}_M^J(\vartheta)$. For $g_J^\lambda(\alpha)$, the following eigenvalue equation is derived:

$$\left\{\frac{D}{2}\left[\frac{\partial^2}{\partial\alpha^2}+2\mathrm{cth}\alpha\frac{\partial}{\partial\alpha}-\frac{J(J+1)}{\mathrm{sh}^2\alpha}\right]-\lambda\right\}g_J^\lambda(\alpha)=0. \quad (25)$$

Here the discrete index J takes the values $J=0,1,2,\ldots$ and $M=-J,-J+1,\ldots,0,1,\ldots,J$. The eigenfunctions for the Laplace-Beltrami operator with the eigenvalues $\lambda=\frac{D}{2}(\varkappa^2+1)$, satisfying the boundary condition, are given by

$$g_J^\varkappa(z)=C_J^\varkappa(z^2-1)^J\left(\frac{d}{dz}\right)^{J+1}\cos(\varkappa\,\mathrm{arch}\,z), \quad (26)$$

with $z=\mathrm{ch}\alpha$ and $C_J^\varkappa=(-1)^{J+1}\prod_{k=0}^J\frac{1}{\sqrt{2\pi}}(\varkappa^2+k^2)^{-1/2}$. The eigenfunctions $\tilde{\Phi}_M^J(\alpha,\vartheta,\varphi,\varkappa)$ satisfy the relations of orthogonality and completeness. The transition probability is determined by the initial condition $\Phi(\alpha,\vartheta,\varphi,\tau=0|\alpha_0,\vartheta_0,\varphi_0,0)=(\mathrm{sh}\alpha)^{-2}(\sin\vartheta)^{-1}\delta(\alpha-\alpha_0)\delta(\vartheta-\vartheta_0)\delta(\varphi-\varphi_0)$. Using the orthogonality relation, we can write

$$\Phi(\alpha,\vartheta,\varphi,\tau|\alpha_0,\vartheta_0,\varphi_0,0)=\sum_{M,J}\int_0^\infty\tilde{\Phi}_M^J(\alpha,\vartheta,\varphi,\varkappa)\tilde{\Phi}_M^{*J}(\alpha_0,\vartheta_0,\varphi_0,\varkappa)\times\exp\left[-\frac{D}{2}(\varkappa^2+1)\tau\right]d\varkappa. \quad (27)$$

Let us now consider the fundamental solution $J=0$. Substituting $g_0^\varkappa(\alpha)=-\frac{1}{\sqrt{2\pi}}(\mathrm{sh}\alpha)^{-1}\sin\varkappa\alpha$ into Eq. (27) gives the transition probability

$$\Phi(\alpha,\tau|\alpha_0,0)=C\left(\tau\frac{D}{2}\right)^{-1/2}\exp\left\{-\tau\frac{D}{2}\right\}\frac{\mathrm{sh}\left(\frac{\alpha\alpha_0}{D\tau}\right)}{\mathrm{sh}\alpha\,\mathrm{sh}\alpha_0}\times\exp\left\{-\frac{\alpha^2+\alpha_0^2}{2D\tau}\right\}, \quad (28)$$

with $C=2(4\pi)^{-3/2}$. For $\alpha_0\to 0$, this solution was first found in [31]. For small velocities ($\alpha\ll 1$), the transition distribution shows a remarkable behavior. If we solve the corresponding nonrelativistic Kramers equation [28], substituting in Eq. (25) $\mathrm{sh}\alpha\to\alpha$, $\mathrm{ch}\alpha\to 1$, we find for $J=0$ $g_0^\varkappa(\alpha)=-\frac{1}{\sqrt{2\pi}}(\alpha)^{-1}\sin\varkappa\alpha$, but the eigenvalue is given by $\lambda=\frac{D}{2}\varkappa^2$ and the solution now is

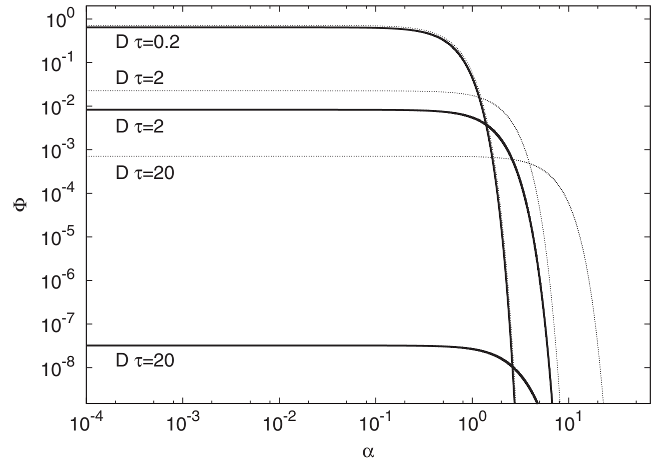

FIG. 1. Relativistic [solid thick lines, Eq. (28)] and nonrelativistic [dotted thin lines, Eq. (29)] distributions for different times $D\tau$ and $\alpha_0=0$.

$$\Phi(a,\tau|a_0,0)=C\left(\frac{D}{2}\tau\right)^{-1/2}\alpha^{-1}\alpha_0^{-1}\left[\exp\left\{-\frac{(\alpha-\alpha_0)^2}{2D\tau}\right\}-\exp\left\{-\frac{(\alpha+\alpha_0)^2}{2D\tau}\right\}\right]. \quad (29)$$

For $\alpha_0=0$, Eq. (29) passes to the known Wiener distribution in the velocity space. In the limit $\alpha\ll 1$, the short-time behavior of Eq. (29) is up to an exponential small factor in agreement with Eq. (28); however, in the long-time behavior both solutions differ by the exponential factor $\exp(-\frac{D}{2}\tau)$. In Fig. 1, the relativistic distribution (28) is presented by the solid lines for $\alpha_0=0$ and the Wiener distribution (29) by the dotted lines. As can be seen, both distributions differ by orders of magnitudes even in the nonrelativistic region $a\ll 1$ for long times $D\tau\gg 1$. This discrepancy can be explained by the topological properties of the hyperbolic space (included by the boundary conditions), which are different from that of the Euclidean space in the nonrelativistic theory. The connection between local and global properties of diffusion processes is a topic in the mathematical field of heat kernels on Riemannian manifolds [32]. The appearance of the factor $\exp(-\frac{D}{2}\tau)$ can also be explained by physical arguments; it comes from that in the hyperbolic coordinates, the Jacobian is proportional to $\mathrm{sh}^2\alpha$, which is exponentially large for $\tau\to\infty$. For large τ, the entire velocity space is explored and the small factor $\exp(-\frac{D}{2}\tau)$ cancels the exponentially large Jacobian and guarantees the probability conservation.

VI. CONCLUSIONS

In conclusion, a theory of Markovian diffusion processes within the framework of the special theory of relativity is formulated. Since the velocity space in relativity is a hyperboloid (or a special three-dimensional Riemannian manifold) embedded into the velocity Minkowski space the mathematical stochastic calculus on Riemannian manifolds is applied here adopted to the velocity space. With the definition of a stochastic differential equation, which generalizes the Langevin equation within the frame of relativity theory and introduces a moving velocity frame, a relativistic diffusion equation is derived. This generalized Kramers equation is invariant under Lorentz transformations. In the case of a relativistic gas in a heat bath with constant friction coefficient, its steady-state solution is identical with the Jüttner distribution. An analytical nonsteady solution for the transition probability is given for the special case of vanishing external fields. This solution differs from the Wiener velocity distribution even for small velocities due to topological reasons.

The formalism presented in this paper can be extended to a theory within the framework of general relativity. This will be done in a forthcoming paper.

APPENDIX

In this appendix, some basic notations, formulas, and theorems of the stochastic calculus on Euclidean and Riemannian manifolds are summarized.

Diffusion processes in a d-dimensional Euclidean space are described by stochastic differential equations of the form [24–27]

$$dX^i = \sigma^i_a(\tau,\mathbf{X})dW^a + b^i(\tau,\mathbf{X})d\tau. \tag{A1}$$

$\mathbf{X}=(X_1,\ldots,X_d)\in R^d$ is a stochastic process with $\mathbf{X}(0)=\mathbf{x}$; $\mathbf{x}=(x_1,\ldots,x_d)$, and τ is the time ($\tau\geq 0$). The diffusion coefficients $\sigma^i_a(\tau,\mathbf{X})$ are given matrices and the drift coefficients $b^i(\tau,\mathbf{X})$ are coefficients of a smooth vector field. W^a's are the components of the elementary Wiener process. Equation (A1) can be transformed into an integral equation

$$X^i_\tau = X^i_0 + \int_0^\tau \sigma^i_a(s,\mathbf{X})dW^a_s + \int_0^\tau b^i(s,\mathbf{X})ds. \tag{A2}$$

The stochastic integral in the second term of Eq. (A2) is defined as the limit $\int_0^\tau \sigma^i_a(s,\mathbf{X})dW^a_s=\Sigma^n_{i=1}\sigma^i_a(s^*_i,\mathbf{X})[W^a(s_i)-W^a(s_{i-1})]$ as $n\to\infty$. This integral depends on the choice of the intermediate point s^*_i. With the choice $s^*_i=s_{i-1}$ (postpoint rule), the Ito stochastic integral is defined. The Ito integral is a Markovian process and plays a fundamental role in the theory of diffusion processes and most of mathematical treatments can only rigorously proven by using this calculus. Alternatively, choosing $s^*_i=s_{i-1}$ (midpoint rule) the Stratonovich stochastic integral is defined. The Stratonovich integral has the advantage of leading to ordinary chain rule formulas under a transformation. This property makes the Stratonovich integral natural to use for stochastic differential equations on Riemannian manifolds. However, in general Stratonovich integrals are not Markovian processes, which hinders rigorous mathematical treatments in most cases. Note that the chosen interpretation has to be denoted in the differential equation. The symbol $\sigma^i_a(\tau,\mathbf{X})dW^a$ implies the Ito integral interpretation and $\sigma^i_a(\tau,\mathbf{X})\circ dW^a_\tau$ the Stratonovich interpretation.

With the Ito interpretation, the solution X^i_τ of Eq. (A1) is denoted as an Ito process if the diffusion and drift coefficients satisfy the Lipschitz condition, and $\sigma^i_a(\tau,\mathbf{X})$ is adapted to the fundamental Wiener process W^a_τ [25]. An Ito process has the important property of being Markovian. Then $\mathbf{Y}_\tau=f(\mathbf{X}_\tau)$ is also an Ito process. Associated to an Ito process is the diffusion generator $\mathbf{A}$ of $\mathbf{X}_\tau$, which is defined to act on a suitable function f by

$$\mathbf{A}f = \lim_{t\to 0}\frac{E^x[f(\mathbf{X}_\tau)]-f(\mathbf{x})}{t}, \tag{A3}$$

where $\mathbf{x}=\mathbf{X}_o$ is the initial point of $\mathbf{X}_\tau$. For the stochastic process described by Eq. (A1), $\mathbf{A}$ is given by [24–27]

$$\mathbf{A}f = \frac{D}{2}\delta^{ab}\sigma^i_a(\tau,\mathbf{X})\sigma^j_b(\tau,\mathbf{X})\partial_i\partial_j f + b^i(\tau,\mathbf{X})\partial_i f. \tag{A4}$$

The generator $\mathbf{A}$ describes how the expected value $u(\tau,\mathbf{x})=E^x[f(\mathbf{X}_\tau)]$ of any smooth function f of $\mathbf{X}$ evolves in time and satisfies the following equation:

$$\frac{\partial}{\partial\tau}u(\tau,\mathbf{x}) = \mathbf{A}u(\tau,\mathbf{x}), \tag{A5}$$

with $u(0,\mathbf{u})=f(\mathbf{x})$. Equation (A5) is denoted as the Kolmogorov's backward equation. The Fokker-Planck equation (or forward Kolmogorov equation) describes how the probability density function $\phi(\tau,\mathbf{x})$ of $\mathbf{X}_\tau$ evolves with time. The probability density function can be used to calculate the expected value $E^x[f(\mathbf{X}_\tau)]$ by $E^x[f(\mathbf{X}_\tau)]=\int_\Omega f(\mathbf{x})\phi(\tau,\mathbf{x})dx_1\cdots dx_d$, where Ω in the domain of the d-dimensional space of the variables $X_{1,\ldots,}X_d$. The Fokker-Planck equation within the Ito integral interpretation is given by the following equation:

$$\frac{\partial}{\partial\tau}\phi(\tau,\mathbf{x}) = A^*\phi(\tau,\mathbf{x}), \tag{A6}$$

with the adjoint operator $\mathbf{A}^*$,

$$\mathbf{A}^*f = \frac{D}{2}\delta^{ab}\partial_i\partial_j\sigma^i_a(\tau,\mathbf{X})\sigma^j_b(\tau,\mathbf{X})f - \partial_i b^i(\tau,\mathbf{X})f. \tag{A7}$$

Since the stochastic calculus on Riemannian manifolds is naturally formulated in the Stratonovich integral interpretation, we will consider the connection between both types of integrals. Let us formulate the stochastic differential equation (A1) with the Ito interpretation by a corresponding equation with the Stratonovich interpretation,

$$dX^i = \tilde{\sigma}^i_a(\tau,\mathbf{X})\circ dW^a + \tilde{b}^i(\tau,\mathbf{X})d\tau. \tag{A8}$$

There exists a connection between Ito and the Stratonovich integrals [24–27], which allows to associate the diffusion and drift terms in one of the interpretations with the other,

$$\tilde{b}^i(\tau,\mathbf{X}) = b^i(\tau,\mathbf{X}) - \delta^{ab}\frac{D}{2}\sigma^j_a(\tau,\mathbf{X})\partial_j\sigma^i_b(\tau,\mathbf{X}),$$

$$\sigma^i_a(\tau,\mathbf{X}) = \tilde{\sigma}^i_a(\tau,X). \tag{A9}$$

Substituting $\tilde{b}^i(\tau,\mathbf{X})$ into Eq. (A4), the diffusion operator $\mathbf{A}$ in the Stratonovich interpretation is

$$\mathbf{A} = \frac{D}{2}\delta^{ab}\sigma^i_a(\tau,\mathbf{x})\partial_i\sigma^j_b(\tau,\mathbf{x})\partial_j + \tilde{b}^i(\tau,\mathbf{x})\partial_i. \quad \text{(A10)}$$

The Fokker-Planck equation (A6) with respect to the Stratonovich interpretation is then given by

$$\frac{\partial}{\partial\tau}\phi(\tau,\mathbf{x}) = \frac{D}{2}\delta^{ab}\partial_i\sigma^i_a(\tau,\mathbf{x})\partial_j[\sigma^j_b(\tau,\mathbf{x})\phi(\tau,x)] - \partial_i\tilde{b}^i(\tau,\mathbf{x})\phi(\tau,\mathbf{x}). \quad \text{(A11)}$$

Introducing the fundamental vector fields

$$L_a = \sigma^i_a(\tau,\mathbf{x})\partial_i, \;\; L_0 = \tilde{b}^i(\tau,\mathbf{x})\partial_i, \quad \text{(A12)}$$

the generator $\mathbf{A}$ of the stochastic process in the Stratonovich interpretation can be expressed by the operators L_a, L_0 as follows:

$$\mathbf{A} = \frac{D}{2}\delta^{ab}L_aL_b + L_0. \quad \text{(A13)}$$

Equation (A13) is an important formula for the calculus on Riemannian manifolds. On a Riemannian manifold, the driving Wiener process W^a of a stochastic differential equation is difficult to handle. In differential geometry for a general d-dimensional Riemannian manifold M^d (with definite metric signature) equipped with a Christoffel connection Γ^j_{ib}, it is possible to lift a smooth curve $c^i(t)$ in M^d to a horizontal curve in the tangent bundle TM (which is endowed with a Euclidean structure) by using the bundles of orthonormal frames $e_a = e^i_a(x)\partial_i (i,a=1,\ldots,d)$. The orthonormal frame bundle $O(M)$ is described by the local coordinates $\{r=(x^i,e^i_j)\}=O(M)$. The infinitesimal motion of a smooth curve $x^i(t)$ in M^d is that of $\gamma^j(t)$ in $O(M)$ described by the ordinary differential equations for a parallel transport,

$$dx^i = e^i_a(\mathbf{x})d\gamma^a,$$

$$de^i_a(\mathbf{x}) = -\Gamma^i_{ml}e^l_a dx^m. \quad \text{(A14)}$$

Here $\delta^{ab}e^i_a(\mathbf{x})e^i_a(\mathbf{x})=g^{ij}$, $\partial_i e^j_a = -\Gamma^j_{ib}e^b_a$ g^{ij} is the Riemannian metric and δ^{ab} is the flat Euclidean metric, where δ^{ab} is the Kronecker symbol. $r^i(t)$ is called the horizontal lift of the curve $x^i(t)$ to the orthonormal frame bundle $O(M)$ and it lies in the Euclidean space R^{d+d^2}. The horizontal curve $\gamma^j(t)$ corresponds uniquely to a smooth curve in the tangent space (which can be identified with an Euclidean space R^d). Correspondingly, a random curve can be defined in the same way as in Eq. (A14) by using the canonical realization of a d-dimensional Wiener process and substituting $d\gamma^a \to dW^a(t)$. Therefore, the stochastic differential equation describing diffusion on a Riemannian manifold is [26,27]

$$dx^i = e^i_a(\tau)\circ dW^a + b^i d\tau,$$

$$de^i_a(\tau) = -\Gamma^i_{ml}e^l_a(\tau)\circ dx^m, \quad \text{(A15)}$$

where the components of an arbitrary tangential vector b^i are additionally introduced for a more general situation with an account of an external force field.

The derivation of the Kolmogorov backward equation with the definition of the diffusion operator $\mathbf{A}_{O(M)}$ can be performed by the same rules, as in Euclidean space in the Stratonovich calculus. Corresponding the definition of the fundamental vector fields L_a and L_0 in Eq. (A12), one can now introduce the fundamental horizontal vector fields H_a and H_0 on $O(M)$ for the extended stochastic differential equation system Eq. (A15),

$$H_a = e^i_a\frac{\partial}{\partial x^i} - \Gamma^i_{ml}(\mathbf{x})e^m_a e^l_b\frac{\partial}{\partial e^i_b},$$

$$H_0 = b^i(\tau,\mathbf{X})\partial_i - \Gamma^i_{ml}e^l_a(\tau)b^m\frac{\partial}{\partial e^i_a}, \quad \text{(A16)}$$

and the operator $\mathbf{A}_{O(M)}$ for the stochastic process in the orthonormal frame bundle is given by

$$\mathbf{A}_{O(M)} = \frac{D}{2}\delta^{ab}H_aH_b + H_0. \quad \text{(A17)}$$

$\mathbf{A}_{O(M)}$ is the horizontal lift of the diffusion generator $\mathbf{A}_M$ on the manifold to the orthonormal frame bundle. Obviously, the projection of a function in $O(M)$ to M with $f(\mathbf{r}) = f(\mathbf{x},0)$, $\mathbf{r}=(x^i,e^i_j)$, satisfies the relation

$$\mathbf{A}_{O(M)}f(\mathbf{r}) = A_M f(\mathbf{x}), \quad \text{(A18)}$$

where $\mathbf{A}_M = \frac{D}{2}\delta^{ab}(e^i_a\partial_i e^j_b\partial_j) + b^i\partial_i = (\frac{D}{2}\Delta_M + b^i\partial_i)$ and $\Delta_M = g^{ij}\partial_i\partial_j + g^{ij}\Gamma^k_{ij}\partial_k$ is the Laplace-Beltrami operator. The generalized Kolmogorov backward equation on a Riemannian manifold is obtained by

$$\frac{\partial}{\partial\tau}u(\tau,\mathbf{x}) = \left(\frac{D}{2}\Delta_M + b^i\partial_i\right)u(\tau,\mathbf{x}). \quad \text{(A19)}$$

The Fokker-Planck operator on a Riemannian manifold cannot be derived directly like in the Euclidean case. But as explained above, this operator is given by the adjoint of the diffusion generator $\mathbf{A}^*$ (which includes the volume element $\sqrt{g}$, $g=\det\{g_{ij}\}$). Since the Laplace-Beltrami operator is self-adjoint $\Delta_M = \Delta^*_M$, the generalized Fokker-Planck equation on a Riemannian manifold takes the form

$$\frac{\partial\Phi}{\partial\tau} = -\operatorname{div}_x(\mathbf{b}\Phi) + \frac{D}{2}\Delta_M\Phi, \quad \text{(A20)}$$

where $\operatorname{div}_x(\mathbf{b}\Phi) = g^{-1/2}\partial_i(g^{1/2}b^i\Phi)$ is the divergence operator in the Riemannian manifold, $\Phi=\Phi(\mathbf{x},\tau|\mathbf{y},0)$ is the transition probability with the initial condition $\Phi(\mathbf{x},0|\mathbf{y},0)=\delta(\mathbf{x}\text{-}\mathbf{y})$ and adequate boundary conditions at infinity. The probability density $\varphi(\mathbf{x},\tau)$ is determined by the same equation with the initial condition $\varphi(\mathbf{x},\tau=0)=\varphi^0(\mathbf{x})$.

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